Introduction to Mathematical Quantum Theory

Text of the Exercises

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Exercise 1

Let V be a Banach space and E a nonempty subset of V such that for any $\xi \in V^*$ there exists a finite constant C_ξ such that

$$\sup_{x \in E} |\xi(x)| \leqslant C_\xi. \tag{1}$$

Prove that E must be bounded.

Hint: Consider the map $J : V \to V^{**}$ defined as

$$[J(x)](\xi) := \xi(x) \qquad \forall x \in V,\ \xi \in V^*. \tag{2}$$

Prove that $\|J(x)\|_{V^{**}} = \|x\|$ for any $x \in V$. Use the Uniform Boundedness Principle to show that $J(E)$ is bounded and conclude.

Exercise 2

Consider (X, Ω) a measurable space (i.e., a set X with a σ-algebra Ω in it), and consider a projection-valued measure with values in $\mathcal{H}$ an Hilbert space. Let $E,\ F \in \Omega$.

- **a** Prove that if $E \cap F = \varnothing$ then $\operatorname{Ran}\mu(E) \perp \operatorname{Ran}\mu(F)$.
- **b** Prove that $\mu(E)\mu(F)$ is an orthogonal projector and that

$$\operatorname{Ran}(\mu(E)\mu(F)) = \operatorname{Ran}\mu(E) \cap \operatorname{Ran}\mu(F). \tag{3}$$

Exercise 3

Let $\mathcal{H}$ be an Hilbert space. Let A be a self-adjoint bounded operator over $\mathcal{H}$. Let B a bounded operator over $\mathcal{H}$ such that $[A, B] = 0$. Consider a bounded complex-valued measurable function f. Prove that $[f(A), B] = 0$.

Exercise 4

Let $\mathcal{H}$ be an Hilbert space. Let T be a bounded operator over $\mathcal{H}$. We proved in class that in general $R(T) \leqslant \|T\|$, where

$$R(T) := \sup_{\lambda \in \sigma(T)} |\lambda|. \tag{4}$$

Exhibit an explicit operator such that $R(T) < \|T\|$.